16th Workshop on Markov Processes and Related Topics
July. 12-16, 2021 Changsha, China

Extinguishing Behaviors for Continuous-State Nonlinear Branching Processes

Junping Li[a] , Yingchun Tang[a] & Xiaowen Zhou[b]
[a]School of Mathematics and Statistics, Central South University, China
[b] Department of Mathematics and Statistics, Concordia University, Canada
1060864320@qq.com

Abstract

A nonnegative continuous-state branching process becomes extinguishing if it converges to 0 but never hits 0 as time goes to infinity. We consider a class of continuous-state nonlinear processes obtained from spectrally positive stable like Lévy processes by Lamperti type time changes using regularly varying (at 0) rate functions, and obtain several large time asymptotic results on the extinguishing behaviors. In particular, we show that, depending on whether the stable index for the spectrally positive Lévy process is smaller than or equal to the regularly varying index for the rate function, a phase transition occurs for the convergence of rescaled first passage times of levels approaching 0. We find conditions on convergence in probability and convergence almost surely, respectively. We also obtain integral tests on almost sure long time fluctuation of the running minimum process.

Introduction

Let Z be a spectrally positive Lévy process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with log-Laplace exponent ψ, i.e. Z is a stochastic process with stationary independent increments and with no negative jumps, and for $\lambda \geq 0$,

$$\mathbb{E}e^{-\lambda(Z_t - Z_0)} = e^{t\psi(\lambda)},$$

where the Laplace exponent

$$\psi(\lambda) = \mu\lambda + \frac{\sigma^2\lambda^2}{2} + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\pi(\mathrm{d}x)$$

for σ-finite Lévy measure π satisfying $\int_0^\infty (x \wedge x^2)\pi(\mathrm{d}x) < \infty$. In this paper we only consider process Z that is not a subordinator.

Let R be a positive locally bounded function on $(0, \infty)$ satisfying $\inf_{x \geq \delta} R(x) > 0$ for any $\delta > 0$. For

$$\tau_b^- := \inf\{t > 0 : Z_t \leq b\}$$

with the convention $\inf \emptyset = \infty$, let

$$\eta_t = \int_0^{t \wedge \tau_0^-} \frac{ds}{R(Z_s)}, \quad t \geq 0$$

and η^{-1} be the right continuous inverse of η. Define $X_t := Z_{\eta^{-1}(t)}$ for all $t \in [0, \eta(\tau_0^-))$ and $X_t := 0$ for all $t \in [\eta(\tau_0^-), \infty)$. Process X is the so-called continuous-state nonlinear branching process with branching mechanism described by ψ and branching rate function R, which was first introduced in Li [3] where the rate function is a power function. Intuitively, it is a generalized continuous-state branching process whose branching rate depends on the current population size, and consequently, it does not have the additive branching property in general. But it remains a Markov process as the time change of a Lévy process. Process X with $R(x) = x$ reduces to the classical continuous-state branching process that has been well studied.

The nonlinear branching mechanism allows more interesting behaviors for the process. In particular, the boundary classification and the associated asymptotic behaviors have been investigated for the continuous-state nonlinear branching processes. The coming down from infinity behaviors for such continuous-state nonlinear branching processes are studied in Foucart et al. [1]. The explosion behaviors for the continuous-state nonlinear branching processes are studied in Li and Zhou [2]. In these two papers, by analyzing the weighted occupation time for process Z, the speeds of coming down from infinity and explosion are identified respectively for rate function R that is either a perturbation of the power function or a perturbation of the exponential function and for process Z that either drifts to infinity or is stable like.

In this paper we continue to investigate the extinguishing behaviors for continuous-state nonlinear branching processes. For any $b \geq 0$ write

$$T_b^- := \inf\{t > 0 : X_t \leq b\}$$

with the convention $\inf \emptyset = \infty$. Let $W(x)$ be the scale function of the associated spectrally negative Lévy process $-Z$; i.e.

$$\int_0^\infty W(x)e^{-\lambda x}\mathrm{d}x = \frac{1}{\psi(\lambda)}, \qquad \lambda > \Phi(0), \tag{1}$$

where $\Phi(0) := \sup\{\lambda \geq 0 : \psi(\lambda) = 0\}$ and $\Phi(0) = 0$ if $\mathbb{E}Z_1 \leq 0$.

Main results

To investigate the extinguishing behaviors for continuous-state nonlinear branching processes, throughout this paper we always assume that

- the spectrally positive Lévy process Z does not drift to ∞, i.e. $\mathbb{E}Z_1 \leq 0$.

Given $1 < \alpha \leq 2$ and $\theta \geq \alpha$, we need some of the following assumptions for the main results.

•

$$R(y) = y^\theta l_1(y) \quad \text{and} \quad W(y) = \frac{y^{\alpha-1}l_2(y)}{\Gamma(\alpha)} \tag{2}$$

for positive functions $l_1(y)$ and $l_2(y)$ on $(0, \infty)$ such that both functions are slowly varying at $0+$, $l_1(y)$ is locally bounded away from 0 and is either slowly varying at ∞ or uniformly bounded away from 0 for y near ∞, and $l_2(y)$ is either slowly varying at ∞ or uniformly bounded away from ∞ for y near ∞.

- for $\theta = \alpha$ and for the same functions $l_1(y)$ and $l_2(y)$ in (2) we assume that

$$\int_{0+} \frac{l_2(y)}{y l_1(y)} dy = \infty. \tag{3}$$

Proposition 0.1. *Suppose that (2) holds for $\theta > \alpha$. Then under $\mathbb{P}_x$ for any $x > 0$, process X becomes extinguishing with probability one, and as $b \to 0+$, $\frac{T_b^-}{m(x,b)}$ converges in distribution to some random variable $S_{\alpha,\theta}$ whose Laplace transform is given by*

$$\mathbb{E}_x\left[e^{-\lambda S_{\alpha,\theta}}\right] = \left[\sum_{n=0}^\infty \left(\frac{\lambda\Gamma(\theta)}{\Gamma(\theta-\alpha)}\right)^n \prod_{i=1}^n \frac{\Gamma(i\theta - i\alpha)}{\Gamma[i\theta - (i-1)\alpha]}\right]^{-1}, \ \lambda > 0. \tag{4}$$

We prove a convergence in probability result for $T_b^-/m(x,b)$ as $b \to 0+$ for the critical case $1 < \alpha = \theta \leq 2$. To this end, we further need the following condition concerning $W(y)$ for large values of y. For $x > 0$ and some $a > 0$,

$$\int_a^\infty \frac{W(y) - W(y-x)}{R(y)} dy \int_a^\infty \frac{W(z) - W(z-y)}{R(z)} dz < \infty. \tag{5}$$

Observe that $m_2(x,a) < \infty$ under condition (5), which holds if $\int_a^\infty \frac{W(y)}{R(y)} dy < \infty$.

Theorem 0.2. *Given $1 < \alpha = \theta \leq 2$ and $x > 0$, suppose that both (2) and (3) hold, and (5) holds. Then process X becomes extinguishing with probability one under $\mathbb{P}_x$. Moreover, $m_2(x,b) < \infty$ for any $x > b > 0$,*

$$m(x,b) \sim \int_b^1 \frac{l_2(y)}{y l_1(y)} dy < \infty \quad \text{as} \quad b \to 0+$$

and

$$\frac{T_b^-}{m(x,b)} \to 1 \quad \text{in } \mathbb{P}_x \text{ as} \quad b \to 0+. \tag{6}$$

Theorem 0.3. *Under the condition of Proposition 0.2, if there exists a decreasing sequence (z_k) such that $z_k \to 0+$ as $k \to \infty$, and in addition,*

$$\sum_{k=1}^\infty \frac{Var_x(T_{z_k}^-)}{m(x,z_k)^2} < \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{m(x,z_{k+1})}{m(x,z_k)} = 1, \tag{7}$$

then

$$\frac{T_b^-}{m(x,b)} \to 1 \quad \mathbb{P}_x\text{-almost surely as} \quad b \to 0+. \tag{8}$$

Proposition 0.4. *Under the condition of Proposition 0.2, we have*

$$Var_x(T_b^-) \leq \left(c^* \int_b^1 \frac{l_2(z)^2}{z l_1(z)^2} dz\right) \vee \left(c \int_b^1 \frac{l_2(z)}{z l_1(z)} dz\right)$$

for constants $c > 0$ and $c^* = \int_0^1 \frac{1-(1-x)^{\alpha-1}}{x} dx > 0$.

Further Research

We will consider the case $\mathbb{E}Z_1 \geq 0$. If $\mathbb{E}Z_1 \geq 0$, continuous-state nonlinear processes may explode or extinct, we focus on the case that the process extinct.

References

[1] P.-S. Li C. Foucart and X. Zhou. Time-changed spectrally positive lévy processes started from infinity. *Bernoulli*, arXiv:1901.10689, 2020.

[2] B. Li and X. Zhou. On the explosion of a class of continuous-state nonlinear branching processes. *Submitted*, arXiv: 1910.05914, 2019.

[3] P.-S. Li. A continuous-state polynomial branching process. *Stochastic Process Application*, 129(8):2941–2967, 2019.